Research Article

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Resonance of Nanoscale Beam due to Various Sources in Modified Couple Stress Thermoelastic Diffusion with Phase Lags

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Abstract: This paper deals with the study of thermoelastic thin beam in a modified couple stress with three-phase-lag thermoelastic diffusion model subjected to thermal and chemical potential sources. The governing equations are derived by using the Euler-Bernoulli beam assumption and eigenvalue approach. The Laplace transform technique is employed to obtain the expressions for displacements, lateral deflection, temperature change, axial stress and chemical potential. A particular type of instantaneous and distributed sources is taken to show the utility of the approach. The general algorithm of the inverse Laplace transform is developed to compute the results numerically. The numerical results are depicted graphically to show the effects of phase lags, with and without energy dissipation on the resulting quantities. Some special cases are given.

Keywords: Modified couple stress theory, thermoelastic thin beam, three-phase-lag, eigenvalue approach, Laplace transform

1 Introduction

The concept of couple stress linear theory of elasticity was introduced by Voigt [1] and this theory was extended by Cosserat and Cosserat [2] Couple stress theory is an extended continuum theory that includes the effects of a couple per unit area on a material volume, in addition to the classical direct and shear forces per unit area. This immediately admits the possibility of asymmetric stress tensor, since shear stress no longer have to be conjugate in order to ensure rotational equilibrium. Toupin [3] derived the associative constitutive equations for finite deformation of perfectly elastic materials. The linearized theory of couple stress elasticity was developed by Mindlin and Tiersten [4]. Sengupta and Ghosh [5, 6] studied the effect of couple stresses on surface waves in elastic media and propagation of waves in an elastic layer using linearized theory of couple stress elasticity.

Yang, Chong, Lam and Tong [7] modified the classical couple stress theory and proposed a modified couple stress model, in which the couple stress tensor is symmetrical and only one material length parameter is needed to capture the size effect that is caused by micro-structure. The bending and vibration of functionally graded microbeams using a new higher order beam theory and the modified couple stress theory was presented by Simsek and Reddy [8]. The significance of using eigenvalue approach is to reduce the problem on vector-matrix differential equation to an algebraic eigenvalue problems. Thus, the solutions for field variables are obtained by determining the eigenvalues and the corresponding eigenvectors. In this approach, the field quantities are directly involved in the formulation of the problem, and as such, the boundary and initial conditions can be applied directly. Kumar, Singh, and Chadha [9] investigated the problem of micropolar thermoelasticity without energy dissipation on the basis of eigenvalue approach.

The five generalizations of the coupled theory were constructed by Hetnarski and Ignaczak [10] which gives a number of important analytical results. The first generalized thermoelasticity theory with one thermal relaxation time was given by Lord and Shulman [11] The second generalization was given by Green and Lindsay [12], who formulated a temperature rate-dependent thermoelasticity with two thermal relaxation times. The third generalization of the coupled theory of thermoelasticity developed by Hetnarski and Ignaczak [13] is known as low

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temperature thermoelasticity. The coupled theory of thermoelasticity discussed by Green and Naghdi [14] is the fourth generalization theory. Tzou [15] proposed to lump the microstructural effects on heat transport mechanism into the delay response in time in the macroscopic formulation and introduced a new generalization of the Fourier law, called dual phase lag heat conduction law with the introduction of two different lags: one is the phase lag of the heat flux and is related to the thermal wave speed and other is the phase lag of the temperature gradient that represents the time constant for electron-lattice equilibrium and is known as the fifth generalization of thermoelasticity. Three-phase-lag thermoelastic model was given by Roychoudhuri [16].

Thermodiffusion is used to describe the processes of thermomechanical treatment of metals (carboning, nitriding steel, etc.) and these processes are thermally activated with their diffusing substances being, nitrogen, carbon and so on. They are accompanied by deformations of the solid. The theory of thermoelastic with mass diffusion was formulated by Nowacki [17]. The coupled thermoelastic model is applied in this theory, which implies infinite speeds of propagation of thermoelastic waves. Sherief, Saleh and Hamza [18] developed the theory of generalized thermoelastic diffusion that predicts finite speeds of propagation for thermoelastic and diffusive waves. Sherief and Saleh [19] worked on the problem of a thermoelastic half space with a permeating substance in contact with the bounding plane in the context of the theory of generalized thermoelastic diffusion with one relaxation time. The basic equations in generalized thermoelastic diffusion for Green Lindsay (GL-model) theory was discussed by Kumar and Kansal [20].

By using the integral transform technique, deformation due to inclined load in generalized thermodiffusive elastic medium was studied by Sharma [21]. Sarkar and Lahiri [22] presented the two-temperature magnetothermoelastic problem in the context of Lord-Shulman (L-S) theory. The basic set of equations was solved by Laplace transform and eigenvalue approach. The problem of thermoelastic damping in a micro-beam resonator using modified couple stress theory was investigated by Rezazadeh, Vahdat, Tayefeh-rezaei, and Cetinkaya [23]. Abouelregal and Zenkour [24] discussed the problem of an axially moving microbeam subjected to sinusoidal pulse heating and an external transverse excitation with one relaxation time by using Laplace transform and also studied the effects of pulse width of thermal vibration, moving speed and the transverse excitation. Kumar and Devi [25] investigated the deformation in a micropolar thermoelastic diffusion medium due to thermal source by using the finite element method in the context of one relaxation time. Effects of Hall current and rotation in a modified couple stress thermoelastic diffusion due to ramp type loading was studied by Reddy, Romanoff and Loya [26]. Chen and Wang [27] used the finite element method to solve the problem of functionally graded circular plates with modified couple stress theory. Zenk and Abouelregal [28] constructed a new model for composite laminated Reddy plate in modified couple-stress theory based on the global local theory. The vibration of functionally graded microbeams in the context of Green-Naghdi thermoelasticity theory (1993) was presented by Kumar [29] Green and Naghdi [30] presented thermoelastic beam due to thermal source in modified couple stress theory.

The present paper is devoted to the study of thermoelastic thin beam in a modified couple stress with three-phase-lag thermoelastic diffusion model. The basic set of equations is solved by applying the Euler-Bernoulli beam assumption and eigenvalue approach. The Laplace transform is applied to obtain the expressions for displacements, lateral deflection, temperature change, axial stress and chemical potential. The physical quantities are computed numerically and depicted graphically to show the effects of phase lags, Green-Naghdi (II) and Green-Naghdi (III) theories.

2 Basic Equations

The constitutive relations, equations of motion, equation of heat conduction and equation of mass diffusion in a modified couple-stress generalized thermoelastic with mass diffusion in the absence of body forces, body couples, heat and mass diffusive sources are given in equations (7) to (31) as follows:

(i) Constitutive Relations

$$t_{ij} = \lambda e\delta_{ij} + 2\mu e_{ij} - \frac{1}{2}e_{kij}m_{lk,l} - \beta_1 T\delta_{ij} - \beta_2 C\delta_{ij}, \quad (1)$$

$$m_{ij} = 2\alpha\chi_{ij}, \quad (2)$$

$$P = -\beta_2 e_{kk} - aT + bC, \quad (3)$$

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad (4)$$

$$\chi_{ij} = \frac{1}{2}\left(\omega_{i,j} + \omega_{j,i}\right), \quad (5)$$

$$\omega_i = \frac{1}{2}e_{ipq}u_{q,p}, \quad (6)$$

(ii) Equations of Motion

$$\left(\lambda+\mu+\frac{\alpha}{4}\Delta\right)\nabla(\nabla.u)+\left(\mu-\frac{\alpha}{4}\Delta\right)\nabla^2 u - \beta_1\nabla T-\beta_2\nabla C=\rho\,\ddot{u}, \tag{7}$$

(iii) Equation of Heat Conduction

$$K^{\star}\left(1+\tau_v\frac{\partial}{\partial t}\right)\nabla^2 T+K\left(1+\tau_T\frac{\partial}{\partial t}\right)\nabla^2\dot{T} = \left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho c_e\ddot{T}+aT_0\ddot{C}+T_0\beta_1\ddot{e}\right), \tag{8}$$

(iv) Equation of Diffusion

$$\left[D^{\star}\left(1+\tilde{\tau}_v\frac{\partial}{\partial t}\right)+D\frac{\partial}{\partial t}\left(1+\tilde{\tau}_T\frac{\partial}{\partial t}\right)\right]\left(-\beta_2\nabla^2 e_{kk}+b\nabla^2 C-a\nabla^2 T\right) = \left(1+\tilde{\tau}_q\frac{\partial}{\partial t}+\frac{\tilde{\tau}_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\ddot{C}, \tag{9}$$

where t_{ij} are the components of stress tensor, λ and μ are material constants, δ_{ij} is Kronecker's delta, e_{ij} are the components of strain tensor, e_{ijk} is alternate tensor, m_{ij} are the components of couple-stress. Here α_t, α_c are the coefficients of linear thermal expansion and diffusion expansion, respectively, T is the temperature change, C is the mass concentration, α is the couple stress parameter, χ_{ij} is symmetric curvature, ω_i is the rotational vector, P is the chemical potential of the material per unit mass, b is the coefficient describing the measure of mass diffusion effects, a is the coefficient describing the measure of thermoelastic diffusion. $u=(u,v,w)$ is the component of displacement vector, ρ is the density, Δ is the Laplacian operator and ∇ is del operator. c_e is the specific heat at constant strain, T_0 is the reference temperature assumed to be such that $T/T_0 << 1$. K is the coefficient of the thermal conductivity, $K^{\star}$ is the material characteristics constant of the theory, c_e is the specific heat at constant strain, D and $D^{\star}$ are the thermoelastic diffusion constants. Here, τ_T, τ_q and τ_v are the phase-lags of the temperature gradient, the heat flux and thermal displacement gradient, respectively; and $\tilde{\tau}_v$, $\tilde{\tau}_T$ and $\tilde{\tau}_q$ are their respective diffusion relaxation times.

3 Formulation of the Problem

Let us consider a homogeneous, isotropic, rectangular modified couple stress thermoelastic-diffusive beam in a Cartesian coordinate system $Oxyz$ for the displacement vector $u(x,y,z,t)=(u,v,w)$ and temperature change T and concentration C with dimensions of length $(0\le x\le L)$, width $(-d/2\le y\le d/2)$ and thickness $(-h/2\le z\le h/2)$, as shown in Figure 1. We define the x-axis along the length of the beam, and the y-axis along the width and z-axes along the thickness, which also represents the axis of material symmetry. Thus, any plane cross-section initially perpendicular to the axis of the beam remains plane and perpendicular to the neutral surface during bending.

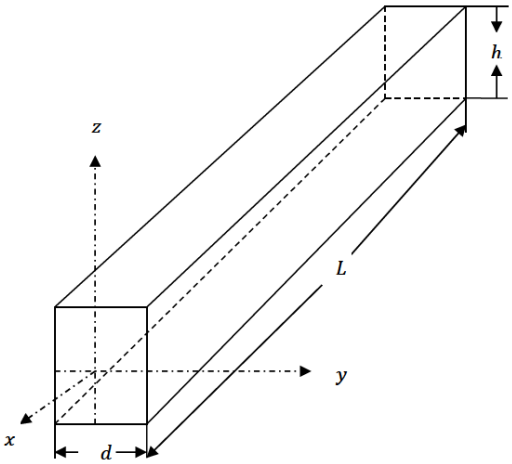

Figure 1: Schematic illustration of the beam set-up

According to the fundamental Euler-Bernoulli assumption for small deflection of a simple bending problem, the displacement components are given by

$$u=-z\frac{\partial w}{\partial x}, v=0, w(x,y,z,t)=w(x,t), \tag{10}$$

where $w(x,t)$ is the lateral deflection of the beam and t is the time. The constitutive relation (1) in one-dimension along the axis and with the help of equation (10), we obtain

$$t_x=-(\lambda+2\mu)z\frac{\partial^2 w}{\partial x^2}-\beta_1 T-\beta_2 C. \tag{11}$$

The bending moment of the cross-section of the beam is given as

$$M=M_{\sigma}+M_m=d\left(\int_{\frac{-h}{2}}^{\frac{h}{2}} t_x z\,dz+\int_{\frac{-h}{2}}^{\frac{h}{2}} m_{xy}dz\right), \tag{12}$$

where M_{σ} and M_m are the components of bending moment due to the classic stress and couple stress tensors, respectively.

Making use of the Euler-Bernoulli assumption (10) and with the aid of equation (11) in equation (12), yields

$$M = -(\lambda + 2\mu) I \frac{\partial^2 w}{\partial x^2} - M_T - M_C - \alpha A \frac{\partial^2 w}{\partial x^2}, \quad (13)$$

Thus, I, M_T and M_C are given by

$$I = \int_{\frac{-h}{2}}^{\frac{h}{2}} dz^2 dz = \frac{dh^3}{12}, M_T = \beta_1 d \int_{\frac{-h}{2}}^{\frac{h}{2}} T z dz, M_C \quad (14)$$
$$= \beta_2 d \int_{\frac{-h}{2}}^{\frac{h}{2}} C z dz,$$

where I is the second moment of the cross-section area of the beam, M_T and M_C are the thermal moment, mass moments of the beam.

The equation of transverse motion of the beam can be written as follows [32]:

$$\frac{\partial^2 M}{\partial x^2} - \rho A \frac{\partial^2 w}{\partial t^2} = 0, \quad (15)$$

where ρ denotes the beam density and $A = dh$ is the cross-sectional area of the beam.

Making use of equation (13) in equation (15), yields

$$((\lambda + 2\mu) I + \alpha A) \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2}{\partial x^2} \left(\beta_1 d \int_{\frac{-h}{2}}^{\frac{h}{2}} T z dz \right) \quad (16)$$
$$+ \frac{\partial^2}{\partial x^2} \left(\beta_2 d \int_{\frac{-h}{2}}^{\frac{h}{2}} C z dz \right) + \rho A \frac{\partial^2 w}{\partial t^2} = 0.$$

For a very thin beam, assuming that the temperature increment and mass concentration varies in terms of $\sin(pz)$ function along the thickness of the beam.

$$T(x, z, t) = T_1(x, t) \sin(pz), \quad (17)$$

where $p = \pi / h$.

For convenience, we define the following dimensionless quantities:

$$(x', z', u', w') = \frac{(x, z, u, w)}{L}, \quad (18)$$
$$(\tau_v', \tau_T', \tau_q', \tilde{\tau}_v', \tilde{\tau}_T', \tilde{\tau}_q', t') = \frac{v}{L} (\tau_v, \tau_T, \tau_q, \tilde{\tau}_v, \tilde{\tau}_T, \tilde{\tau}_q, t),$$
$$(T_1', C_1', t_x') = \frac{(\beta_1 T_1, \beta_2 C_1, t_x)}{E},$$
$$(M', M_T', M_C') = \frac{(M, M_T, M_C)}{dEh^2}, \quad v^2 = \frac{E}{\rho},$$
$$K^* = \frac{c_e (\lambda + 2\mu)}{4}, \quad D = \frac{D^* L}{v}.$$

Using equations (10) and (14) in equations (8) and (9) and then multiplying the resulting equations by $z\,dz$ and integrate over the interval $(-h/2, h/2)$, yield the simplification equations. These equations and also equation (16), with the aid of equations (17) and (18), after dropping the dashes for convenience, can be written as:

$$\frac{\partial^4 w}{\partial x^4} + a_1 \left[\left(\frac{\partial^2 T_1}{\partial x^2} \right) + \left(\frac{\partial^2 C_1}{\partial x^2} \right) \right] + a_2 \frac{\partial^2 w}{\partial t^2} = 0, \quad (19)$$

$$\left[\left(1 + \tau_v \frac{\partial}{\partial t} \right) + a_3 \frac{\partial}{\partial t} \left(1 + \tau_T \frac{\partial}{\partial t} \right) \right] \quad (20)$$
$$\left(\frac{\partial^2 T_1}{\partial x^2} - a_4 T_1 \right) = \left(1 + \tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2}{\partial t^2} \right)$$
$$\left[a_5 \frac{\partial^2 T_1}{\partial t^2} + a_6 \frac{\partial^2 C_1}{\partial x^2} - a_7 \frac{\partial^4 w}{\partial x^2 \partial t^2} \right],$$

$$\left[\left(1 + \tilde{\tau}_v \frac{\partial}{\partial t} \right) + a_8 \frac{\partial}{\partial t} \left(1 + \tilde{\tau}_T \frac{\partial}{\partial t} \right) \right] \quad (21)$$
$$\left[a_9 \frac{\partial^4 w}{\partial x^4} + \left(\frac{\partial^2 C_1}{\partial x^2} - a_4 C_1 \right) - a_{10} \left(\frac{\partial^2 T_1}{\partial x^2} - a_4 T_1 \right) \right]$$
$$= a_{11} \left(1 + \tilde{\tau}_q \frac{\partial}{\partial t} + \frac{\tilde{\tau}_q^2}{2} \frac{\partial^2}{\partial t^2} \right) \frac{\partial^2 C_1}{\partial t^2},$$

Where,

$$a_1 = \frac{2dEL}{p^2 [(\lambda + 2\mu) I + \alpha A]}, \quad a_2 = \frac{\rho A v^2 L^2}{[(\lambda + 2\mu) I + \alpha A]},$$
$$a_3 = \frac{Kv}{K^* L}, \quad a_4 = p^2 L^2, \quad a_5 = \frac{\rho c_e v^2}{K^*}, \quad a_6 = \frac{a T_0 v^2 \beta_1}{\beta_2 K^*},$$
$$a_7 = \frac{\beta_1^2 T_0 p^2 h^3 v^2}{24 E L K^*}, \quad a_8 = \frac{Dv}{D^* L}, \quad a_9 = \frac{-\beta_2^2 p^2 h^3}{24 b E L},$$
$$a_{10} = \frac{a \beta_2}{b \beta_1}, \quad a_{11} = \frac{v^2}{D^* b}.$$

4 Solution of the Problem

We define the Laplace transform as:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \bar{f}(s). \quad (22)$$

where s is the Laplace transform parameter.

The system of equations (19) to (21), after applying the Laplace transform can be written in a matrix form as:

$$DV(x,s) = AV(x,s), \quad (23)$$

where,

$$V = \begin{bmatrix} U \\ DU \end{bmatrix}, \quad U = \begin{bmatrix} \overline{w} & \overline{v} & \overline{T}_1 & \overline{C}_1 \end{bmatrix}^T, \quad (24)$$

$$A = \begin{bmatrix} O & I \\ A_1 & O \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

Here, I denotes the identity matrix of order 4, O denotes the null matrix of order 4 and $[]^T$ is the transpose of matrix, $D = \frac{d}{dz}$.

where,

$$\overline{\tau}_v = (1+\tau_v s), \quad \overline{\tau}_T = s(1+\tau_T s),$$
$$\overline{\tau}_q = \left(1+\tau_q s+\frac{\tau_q^2}{2}s^2\right), \quad \overline{\tilde{\tau}}_v = (1+\tilde{\tau}_v s),$$
$$\overline{\tilde{\tau}}_T = s(1+\tilde{\tau}_T s), \quad \overline{\tilde{\tau}}_q = \left(1+\tilde{\tau}_q s+\frac{\tilde{\tau}_q^2}{2}s^2\right)$$

$$a_{21} = -\left\{a_1(a_{21}+a_{31})+a_2 s^2\right\},$$
$$a_{22} = -\{a_1(a_{22}+a_{32})\}, \quad a_{23} = -\{a_1(a_{23}+a_{33})\},$$
$$a_{24} = -\{a_1(a_{24}+a_{34})\}, \quad a_{31} = \frac{a_6 a_{41}\overline{\tau}_q}{(\overline{\tau}_v+a_3\overline{\tau}_T)},$$
$$a_{32} = \frac{\left(a_6 a_{22}-a_7 s^2\right)\overline{\tau}_q}{(\overline{\tau}_v+a_3\overline{\tau}_T)},$$
$$a_{33} = \frac{a_4(\overline{\tau}_v+a_3\overline{\tau}_T)+\left(a_6 a_{23}+a_5 s^2\right)\overline{\tau}_q}{(\overline{\tau}_v+a_3\overline{\tau}_T)},$$
$$a_{34} = \frac{a_6 a_{42}\overline{\tau}_q}{(\overline{\tau}_v+a_3\overline{\tau}_T)},$$
$$a_{41} = \frac{\left(a_2 a_9 s^2\right)(\overline{\tau}_v+a_3\overline{\tau}_T)}{(1-a_1 a_9)(\overline{\tau}_v+a_3\overline{\tau}_T)-a_6\overline{\tau}_q(a_1 a_9+a_{10})},$$
$$a_{42} = \frac{-a_7 s^2\overline{\tau}_q(a_1 a_9+a_{10})}{(1-a_1 a_9)(\overline{\tau}_v+a_3\overline{\tau}_T)-a_6\overline{\tau}_q(a_1 a_9+a_{10})},$$
$$a_{43} = \frac{a_4(a_1 a_9+a_{10})(\overline{\tau}_v+a_3\overline{\tau}_T)+a_5 s^2\overline{\tau}_q(a_1 a_9+a_{10})}{(1-a_1 a_9)(\overline{\tau}_v+a_3\overline{\tau}_T)-a_6\overline{\tau}_q(a_1 a_9+a_{10})},$$
$$a_{44} = \frac{\left(a_4\left(\overline{\tilde{\tau}}_v+a_8\overline{\tilde{\tau}}_T\right)+a_{11}s^2\overline{\tilde{\tau}_q}\right)(\overline{\tau}_v+a_3\overline{\tau}_T)}{[(1-a_1 a_9)(\overline{\tau}_v+a_3\overline{\tau}_T)-a_6\overline{\tau}_q(a_1 a_9+a_{10})]\left(\overline{\tilde{\tau}}_v+a_8\overline{\tilde{\tau}}_T\right)}.$$

To solve the equation (23) by the eigenvalue approach as [33]. The characteristic equation of the matrix A is written as:

$$\lambda^8 - D_1\lambda^6 + D_2\lambda^4 - D_3\lambda^2 + D_4 = 0. \tag{25}$$

where,

$$D_1 = a_{22}+a_{33}+a_{44},$$
$$D_2 = (a_{22}+a_{33})a_{44}+a_{22}a_{33}-a_{34}a_{43}-a_{24}a_{42}-a_{23}a_{32}-a_{21},$$
$$D_3 = a_{22}(a_{34}a_{43}-a_{33}a_{44})+a_{23}(a_{32}a_{44}-a_{34}a_{42})-a_{24}(a_{32}a_{43}-a_{33}a_{42})-a_{21}(a_{33}+a_{44})+(a_{24}a_{41}+a_{23}a_{31}),$$
$$D_4 = a_{21}(a_{34}a_{43}-a_{33}a_{44})+a_{23}(a_{31}a_{44}-a_{34}a_{41})+a_{24}(a_{33}a_{41}-a_{31}a_{43}).$$

The characteristic roots of the equation (25), which are also the eigenvalues of the matrix A. The eigenvectors $X(x,s)$ corresponding to eigenvalue λ_r can be determined by solving the homogeneous equations:

$$[A-\lambda I]X(x,s) = 0, \tag{26}$$

The set of eigen vectors $X_r(x,s)$ may be obtained as:

$$X_r(x,s) = \begin{bmatrix}X_{r1}(x,s)\\X_{r2}(x,s)\end{bmatrix},\ X_{r1}(x,s) = \begin{bmatrix}b_r\\c_r\\d_r\\q_r\end{bmatrix},\ X_{r2}(x,s) = \lambda_r X_{r1}(x,s)$$

for $\lambda = \lambda_r$, $r = 1, 2, 3, 4$ and

$$X_j(x,s) = \begin{bmatrix}X_{j1}(x,s)\\X_{j2}(x,s)\end{bmatrix},\ X_{j1}(x,s) = \begin{bmatrix}b_r\\c_r\\d_r\\q_r\end{bmatrix},\ X_{j2}(x,s) = \lambda_j X_{j1}(x,s)$$

for $j = r+4$, $\lambda = -\lambda_r$, $r = 1, 2, 3, 4$,

$$b_r = \frac{(\lambda_r^2-a_{22})\delta_4 - a_{23}\delta_2 + a_{24}\delta_1}{a_{21}},$$
$$c_r = \delta_4, d_r = \delta_2, q_r = -\delta_3.$$

$$\delta_1 = \left[\left(a_{31}a_{23}-a_{21}\left(a_{33}-\lambda_r^2\right)\right)(a_{41}a_{32}-a_{31}a_{42}) + \left(a_{31}a_{43}-a_{41}\left(a_{33}-\lambda_r^2\right)\right)\left(a_{31}\left(a_{22}-\lambda_r^2\right)-a_{21}a_{32}\right)\right],$$

$$\delta_2 = \left[(a_{31}a_{24}-a_{21}a_{34})(a_{41}a_{32}-a_{31}a_{42}) - \left(a_{34}a_{41}-a_{31}\left(a_{44}-\lambda_r^2\right)\right)\left(a_{31}\left(a_{22}-\lambda_r^2\right)-a_{21}a_{32}\right)\right],$$

$$\delta_3 = \left[\left(a_{31}\left(a_{22}-\lambda_r^2\right)-a_{21}a_{32}\right)\left(a_{41}\left(a_{33}-\lambda_r^2\right)-a_{31}a_{43}\right)+(a_{31}a_{42}-a_{41}a_{32})\left(a_{31}a_{23}-a_{21}\left(a_{33}-\lambda_r^2\right)\right)\right],$$

$$\delta_4 = \left[(a_{31}a_{24}-a_{21}a_{34})\left(a_{41}\left(a_{33}-\lambda_r^2\right)-a_{31}a_{43}\right) + \left(a_{31}\left(a_{44}-\lambda_r^2\right)-a_{34}a_{41}\right)\left(a_{31}a_{23}-a_{21}\left(a_{33}-\lambda_r^2\right)\right)\right].$$

Thus, the physical quantities are given by:

$$\left(\overline{w},\overline{v},\overline{T}_1,\overline{C}_1\right)(x,s) = \sum_{r=1}^{4}(b_r,c_r,d_r,q_r)C_r e^{-\lambda_r x} \tag{27}$$

$$+\sum_{j=1}^{4}\left(b_{j+4},c_{j+4},d_{j+4},q_{j+4}\right)C_{j+4}e^{\lambda_j x},$$

$$\overline{P}(x,s)= \tag{28}$$
$$\sum_{r=1}^{4}\left\{a_{12}\lambda_r^2 b_r-\sin(pz)\left(a_{13}d_r-a_{14}q_r\right)\right\}C_r e^{-\lambda_r x}+$$
$$\sum_{j=1}^{4}\left\{a_{12}\lambda_j^2 b_{j+4}-sin(pz)\left(a_{13}d_{j+4}-a_{14}q_{j+4}\right)\right\}C_{j+4}e^{\lambda_j x},$$

$$\bar{t}_x(x,s)= \tag{29}$$
$$-\left(\sum_{r=1}^{4}\left\{(\lambda+2\mu)z\lambda_r^2 b_r+\beta_1 d_r+\beta_2 q_r\right\}C_r e^{-\lambda_r x}\right.$$
$$\left.+\sum_{j=1}^{4}\left\{(\lambda+2\mu)z\lambda_j^2 b_{j+4}+\beta_1 d_{j+4}+\beta_2 q_{j+4}\right\}C_{j+4}e^{\lambda_j x}\right),$$

where $a_{12}=\frac{\beta_2 z}{bC_0}$, $a_{13}=\frac{aE}{\beta_1 bC_0}$, $a_{14}=\frac{E}{C_0\beta_2}$, C_1, C_2, C_3, C_4, C_5, C_6, C_7 and C_8 are arbitrary constants.

5 Initial and Boundary Conditions

The initial and boundary conditions should be considered to solve the problem. The initial conditions of the problem are taken in the form as:

$$w(x,t)|_{t=0}=\left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0}=0, \tag{30}$$
$$T(x,t)|_{t=0}=\left.\frac{\partial T(x,t)}{\partial t}\right|_{t=0}=0,$$
$$C(x,t)|_{t=0}=\left.\frac{\partial C(x,t)}{\partial t}\right|_{t=0}=0.$$

Let us consider a nanobeam where both ends are simply supported:

$$w(0,t)=0,\quad \frac{\partial^2 w(0,t)}{\partial x^2}=0,\quad w(\mathrm{L},t)=0, \tag{31}$$
$$\frac{\partial^2 w(L,t)}{\partial x^2}=0.$$

(i) Thermal source

$$T(0,t)=f_1(t), \tag{32}$$

$$P(0,t)=0, \tag{33}$$

(ii) Chemical potential source

$$T(0,t)=0, \tag{34}$$

$$P(0,t)=f_2(t), \tag{35}$$

We also assume that the other side of nanobeam $P(0,t)=f_2(t)$, is thermally insulated and this means that the following relation will be satisfied:

$$\frac{dT(L,t)}{dx}=0,\frac{dC(L,t)}{dx}=0. \tag{36}$$

Applications:

(i) For instantaneous sources

$$f_1(t)=f_2(t)=\delta(t). \tag{37}$$

(ii) For uniformly distributed sources

$$f_1(t)=f_2(t)=\begin{cases}1 & if\ |t|\le a\\ 0 & if\ |t|>a,\end{cases} \tag{38}$$

Applying Laplace transform on equations (31) to (38) and using the value of $\overline{w}$, $\overline{T}_1$, $\overline{C}_1$, $\bar{t}_x$ and $\overline{P}$ from equations (27) to (29), after some simplifications, we obtain

$$\left(\overline{u},\overline{w},\overline{T}_1\right)(x,s)=\sum_{r=1}^{4}\left((z\lambda_r+1)b_r+d_r\right)C_r e^{-\lambda_r x} \tag{39}$$
$$-\sum_{j=1}^{4}\left(\left(z\lambda_{j+4}+1\right)b_{j+4}+d_{j+4}\right)C_{j+4}e^{\lambda_j x},$$

$$\bar{t}_x(x,s)=-\left(\sum_{r=1}^{4}M_r C_r e^{-\lambda_r x}+\sum_{j=1}^{4}M_{j+4}C_{j+4}e^{\lambda_i x}\right), \tag{40}$$

$$\overline{P}(x,s)=\sum_{r=1}^{4}N_r C_r e^{-\lambda_r x}+\sum_{j=1}^{4}N_{j+4}C_{j+4}e^{\lambda_j x}, \tag{41}$$

where,

$$C_1\frac{\Delta_1}{\Delta},\quad C_2\frac{\Delta_2}{\Delta},\quad C_3\frac{\Delta_3}{\Delta},\quad C_4\frac{\Delta_4}{\Delta},\quad C_5\frac{\Delta_5}{\Delta},\quad C_6\frac{\Delta_5}{\Delta},$$
$$C_7\frac{\Delta_7}{\Delta},\quad C_8\frac{\Delta_6}{\Delta},$$

and

$$\sum_{r=1}^{4}M_r=\left\{(\lambda+2\mu)z\lambda_r^2 b_r+\beta_1 d_r+\beta_2 q_r\right\},$$

$$\sum_{j=1}^{4}M_{j+4}=\left\{(\lambda+2\mu)z\lambda_j^2 b_{j+4}+\beta_1 d_{j+4}+\beta_2 q_{j+4}\right\},$$

$$\sum_{r=1}^{4}N_r=\left\{a_{12}\lambda_r^2 b_r-sin(pz)\left(a_{13}d_r-a_{14}q_r\right)\right\},$$

$$\sum_{j=1}^{4} N_{j+4} = \left\{ a_{12}\lambda_j^2 b_{j+4} - sin\,(pz)\left(a_{13}d_{j+4} - a_{14}q_{j+4}\right)\right\}.$$

$$\Delta = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ b_1 e^{-\lambda_1} & b_2 e^{-\lambda_2} & b_3 e^{-\lambda_3} & b_4 e^{-\lambda_4} \\ b_1\lambda_1^2 & b_2\lambda_2^2 & b_3\lambda_3^2 & b_4\lambda_4^2 \\ b_1\lambda_1^2 e^{-\lambda_1} & b_2\lambda_2^2 e^{-\lambda_2} & b_3\lambda_3^2 e^{-\lambda_3} & b_4\lambda_4^2 e^{-\lambda_4} \\ d_1 & d_2 & d_3 & d_4 \\ -d_1\lambda_1 e^{-\lambda_1} & -d_2\lambda_2 e^{-\lambda_2} & -d_3\lambda_3 e^{-\lambda_3} & -d_4\lambda_4 e^{-\lambda_4} \\ N_1 & N_2 & N_3 & N_4 \\ -q_1\lambda_1 e^{-\lambda_1} & -q_2\lambda_2 e^{-\lambda_2} & -q_3\lambda_3 e^{-\lambda_3} & -q_4\lambda_4 e^{-\lambda_4} \end{bmatrix}$$

$$\begin{bmatrix} \cdots & b_5 & b_6 & b_7 & b_8 \\ \cdots & b_5 e^{\lambda_1} & b_6 e^{\lambda_2} & b_7 e^{\lambda_3} & b_8 e^{\lambda_4} \\ \cdots & b_5\lambda_1^2 & b_6\lambda_2^2 & b_7\lambda_3^2 & b_8\lambda_4^2 \\ \cdots & b_5\lambda_1^2 e^{\lambda_1} & b_6\lambda_2^2 e^{\lambda_2} & b_7\lambda_3^2 e^{\lambda_3} & b_8\lambda_4^2 e^{\lambda_4} \\ \cdots & d_5 & d_6 & d_7 & d_8 \\ \cdots & d_5\lambda_1 e^{\lambda_1} & d_6\lambda_2 e^{\lambda_2} & d_7\lambda_7 e^{\lambda_3} & d_8\lambda_8 e^{\lambda_4} \\ \cdots & N_5 & N_6 & N_7 & N_8 \\ \cdots & q_5\lambda_1 e^{\lambda_1} & q_6\lambda_2 e^{\lambda_2} & q_7\lambda_3 e^{\lambda_3} & q_8\lambda_4 e^{\lambda_4} \end{bmatrix}$$

$\Delta_i\,(i = 1, \ldots\ldots\ldots\ldots, 8)$ is obtained by replacing 1^{st}, 2^{nd}, 3^{rd}, 4^{th}, 5^{th}, 6^{th}, 7^{th} and 8^{th} column by $\left[0, 0, 0, 0, \bar{f}\,(s), 0, 0, 0\right]^T$ in for (Thermal source) and $\left[0, 0, 0, 0, 0, \bar{f}\,(s), 0, 0\right]^T$ in for (Chemical potential source)

where,

$$\bar{f}\,(s) = 1, \quad \text{instantaneous source}$$

$$\bar{f}\,(s) = \frac{1 - e^{-s}}{s}, \quad \text{distributed source}$$

6 Particular Cases

6.1 GN-II model

If $K = \tau_v = \tau_T = \tau_q = \tau_q^2 = 0$, in equations (39) to (41), we obtain the corresponding results for displacement component, lateral deflection, temperature change, chemical potential and axial stress of the beam in a modified couple stress thermoelastic diffusion for Green-Naghdi-II (GN-II) model of thermoelasticity.

6.2 GN-III model

In the absence of $\tau_v = \tau_T = \tau_q = \tau_q^2 = 0$, in equations (39) to (41), we obtain the corresponding expressions for displacement component, lateral deflection, temperature change, chemical potential and axial stress of the beam in a modified couple stress thermoelastic materials with energy dissipation in context for GN-III theory of thermoelasticity.

6.3

If we take in equations (39) to (41), we obtain the corresponding results for displacement component, lateral deflection, temperature change, chemical potential and axial stress of the beam for thermoelastic diffusion with three-phase-lag model of thermoelasticity. When we take$D = D^{\star} = C = 0$, in equations (39) to (41), our results as a special case are similar as obtained by Honig and Hirdes [34].

7 Inversion of the Laplace Transform

We obtained the solutions for the displacement component, lateral deflection, temperature change, axial stress and chemical potential in the Laplace transform domain (x,s). We shall now outline briefly the numerical inversion method used to find the solution in the physical domain. Let $\bar{f}\,(s)$ be the Laplace transform of a function $f\,(t)$. The inversion formula of Laplace transform can be written as:

$$\bar{f}\,(s) = L\left[f\,(t)\right] = \int_0^{\infty} e^{-st} f\,(t)dt, \tag{42}$$

$$f\,(t) = L^{-1}\left[\bar{f}\,(s)\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}\,(s)ds, \tag{43}$$

with $s = c + ig$;

where c is an arbitrary real number greater than all the real parts of the singularities of $\bar{f}\,(s)$. We adopt a numerical inversion method on the Fourier series expansion, by which the integral (43) can be approximated as a series

$$\begin{aligned} f\,(t) = \frac{e^{ct}}{t_1}\Bigg[& -\frac{1}{2}\mathrm{Re}\bar{f}\,(c) \\ & + \sum_{j=0}^{\infty}\mathrm{Re}\left(\bar{f}\left(c + \frac{ij\pi}{t_1}\right)\right)\cos\left(\frac{j\pi}{t_1}\right) \\ & - \sum_{j=0}^{\infty}\mathrm{Im}\left(\bar{f}\left(c + \frac{ij\pi}{t_1}\right)\right)\sin\left(\frac{j\pi}{t_1}\right)\Bigg] \\ & - \sum_{j=1}^{\infty} \mathrm{e}^{-2cjt_1} f\,(2jt_1 + t)\,. \end{aligned} \tag{44}$$

for $0 \le t \le 2t_1$. The above series (44) is called the Durbin formula and the last term of this series is called the discretization error. Sur and Kanoria [35] developed a method for accelerating the convergence of the Fourier series and a procedure that computes approximately the best choice of the free parameters.

8 Numerical Results and Discussion

For numerical computations, we have chosen the copper material. The physical data for which are given by Sherief, Saleh and Hamza [18].
$E = 120\text{GPa}$, $\nu = 0.34$, $T_0 = 0.293 \times 10^3\text{K}$, $\rho = 8.954 \times 10^3\text{Kg m}^{-3}$, $K = 0.386 \times 10^3\text{W m}^{-1}\text{ K}^{-1}$, $\alpha_t = 1.78 \times 10^{-5}\text{ K}^{-1}$, $\alpha_c = 1.98 \times 10^{-4}\text{ m}^3\text{ K g}^{-1}$, $c_e = 0.3831 \times 10^3\text{JK g}^{-1}\text{ K}^{-1}$, $a = 2.5\text{Kgm s}^{-2}$, $D = 0.85 \times 10^{-8}\text{ Kgs m}^{-3}$, $a = 1.02 \times 10^4\text{m}^2\text{ s}^{-2}\text{ K}^{-1}$, $b = 9 \times 10^5\text{K g}^{-1}\text{ m}^5\text{ s}^{-2}$, $t = 1.0\text{s}$, $C_0 = 1$, $\tau_\nu = 0.02\text{s}$, $\tau_T = 0.03\text{s}$, $\tau_q = 0.04\text{s}$, $\tilde{\tau}_\nu = 0.015\text{s}$, $\tilde{\tau}_T = 0.01\text{s}$, $\tilde{\tau}_q = 0.1\text{s}$, $L = 1$, $d = 1$, $h = 10$.

MATLAB software has been used for numerical computations. Using this software, the displacement component, lateral deflection, temperature change, axial stress and chemical potential for different theories of thermoelasticity with respect to the length are computed numerically and shown graphically in Figures 2–21. In these figures, small dash line corresponds to three-phase-lag model (TPL), small dash line with centre symbol (– * – * –) corresponds to Green-Naghdi model (GN-II) and dash line with centre symbol (– * –) corresponds to Green-Naghdi model (GN-III) model, respectively.

Instantaneous Source

(*i*) Thermal Source
(*ii*) Chemical Potential Source

Distributed Source

(*i*) Thermal Source
(*ii*) Chemical Potential Source

Figure 2 represents the temperature change with respect to the length for different thermoelasticity theories. It is noticed that the value of temperature decreases initially for the range $0 \le x < 2$, then it increases to attain maximum value and after that it decreases in the remaining region. The value of temperature change is less for the range $0 \le x < 2.5$ for three-phase-lag than that of Green-Naghdi-II and Green-Naghdi-III but a reversed behaviour is seen in the remaining range of length. Figure 3 depicts the lateral deflection with respect to the length for different thermoelasticity theories. It is noticed that the value of lateral deflection decreases for the range and increases for $1 \le x < 4$ and then decreases in the remaining range. The value of lateral deflection is more initially for Green-Naghdi-II as compared with Green-Naghdi-III and three-phase-lag, but opposite behaviour is observed for the considered range.

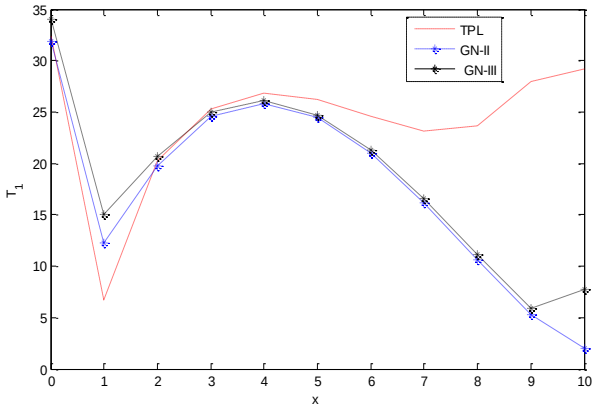

Figure 2: Temperature change with respect to the length for different theories of thermoelasticity

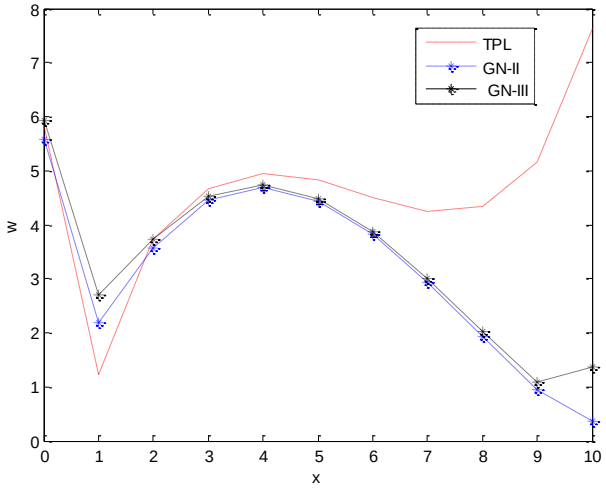

Figure 3: Lateral deflection with respect to the length for different theories of thermoelasticity

Figure 4 shows the displacement component with respect to the length for different thermoelasticity theories. The value of displacement component is stable for the range and then increases in the remaining range for all the cases of the three-phase-lag, GN-II and GN-III. Figures 5 and 6 represent the axial stress and chemical potential with respect to the length for different theories of thermoelasticity. The variation of axial stress and chemical po-

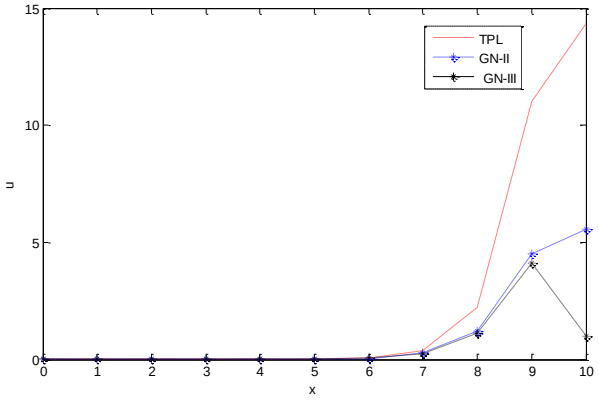

Figure 4: Displacement component with respect to the length for different theories of thermoelasticity

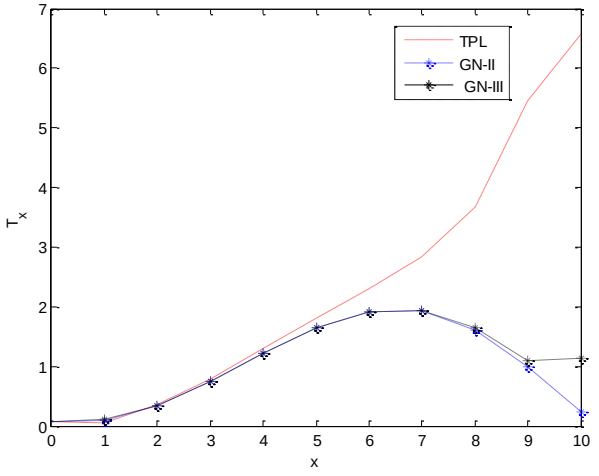

Figure 5: Axial stress with respect to the length for different theories of thermoelasticity

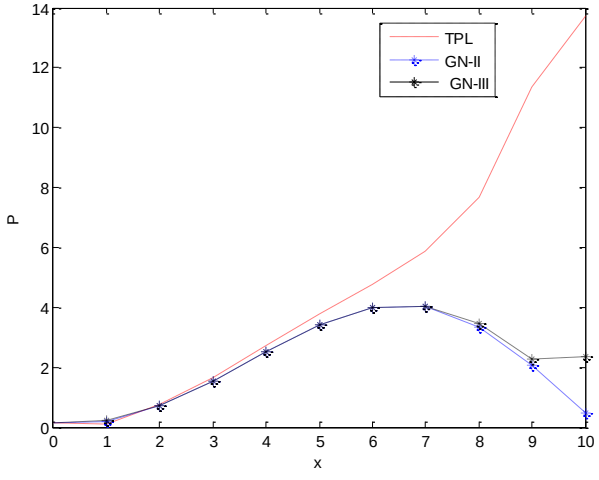

Figure 6: Chemical potential with respect to the length for different theories of thermoelasticity

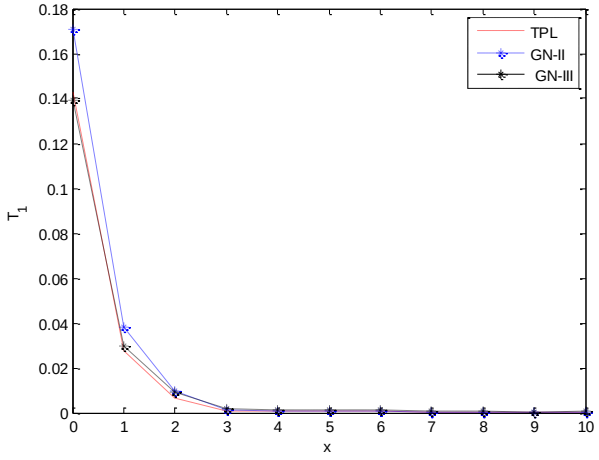

Figure 7: Temperature change with respect to the length for different theories of thermoelasticity

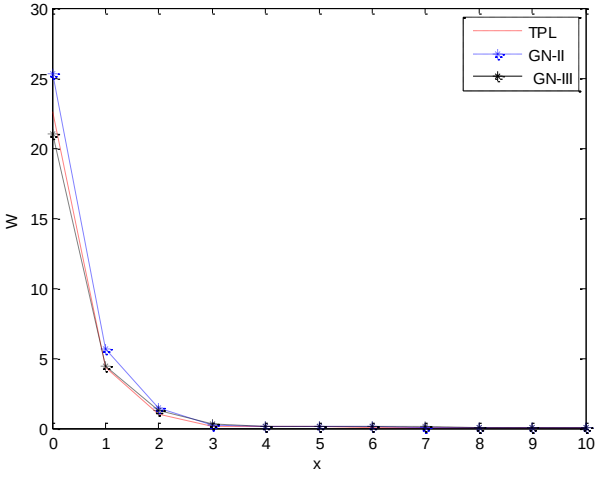

Figure 8: Lateral deflection with respect to the length for different theories of thermoelasticity

tential are different but behaviour is the same for all the cases of three-phase-lag, GN-II and GN-III. The value of axial stress and chemical potential increases monotonically with respect to the length for three-phase-lag, whereas its value increases for the range $0 \leq x \leq 7$ and then decreases.

Figure 7 depicts the temperature change with respect to the length for different theories of thermoelasticity. The behaviour and variation of temperature is the same in all the theories, that is, three-phase-lag, GN-II and GN-III. It is clear from the figure that the value of temperature decreases monotonically with the increase in length for all the theories. Figure 8 shows the lateral deflection with respect to the length for different theories of thermoelasticity. It is evident that the value of lateral deflection initially decreases for the range $0 \leq x \leq 6$ and then its value remains stationary for the remaining range. The values of lat-

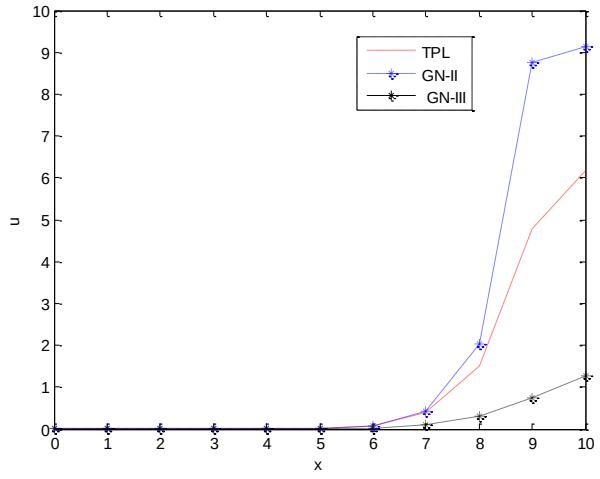

Figure 9: Displacement component with respect to the length for different theories of thermoelasticity

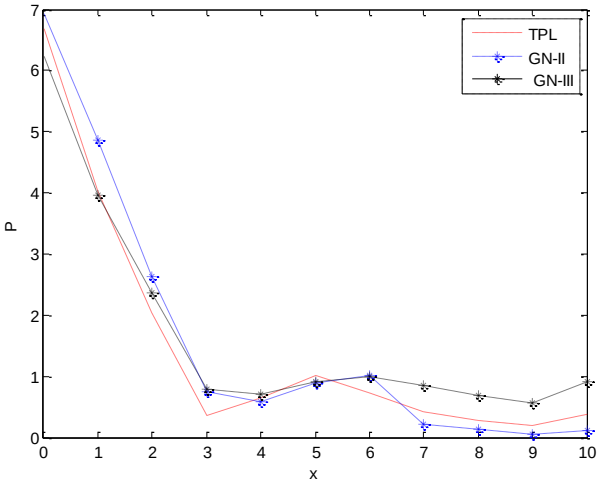

Figure 11: Chemical potential with respect to the length for different theories of thermoelasticity

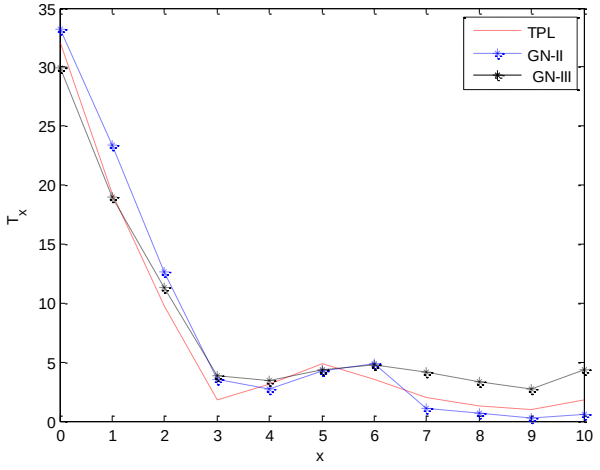

Figure 10: Axial stress with respect to the length for different theories of thermoelasticity

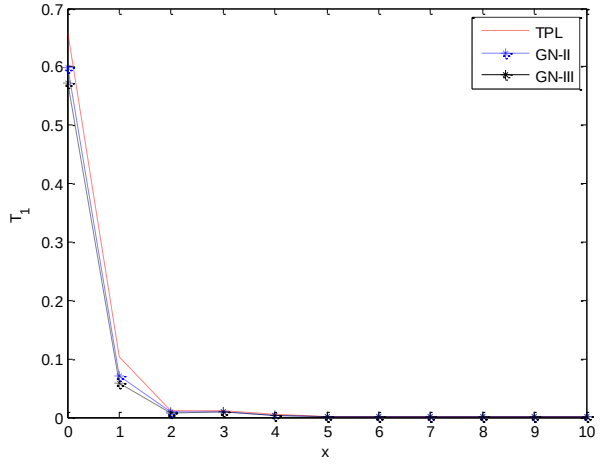

Figure 12: Temperature change with respect to the length for different theories of thermoelasticity

eral deflection is higher for GN-II in comparison with three-phase-lag (TPL) and GN-III. Figure 9 represents the displacement component with respect to the length for different thermoelasticity theories. The value of displacement component is stable initially for the range $0 \leq x \leq 6$, and then, it increases in the assumed region for all the cases of thermoelasticity. The value of displacement component is greater for the three-phase-lag than that for GN-II and GN-III.

Figures 10 and 11 represent the axial stress and chemical potential with respect to the length for different thermoelasticity theories. The behaviour and variation of axial stress and chemical potential are same for all theories. Also, the value of axial stress and chemical potential decreases for the range and remains oscillatory in the considered region of length for three-phase-lag, GN-II and GN-III. The value of axial stress and chemical potential for GN-II and GN-III are smaller in comparison with the three-phase-lag. Figures 12 and 13 depict the temperature change and lateral deflection with respect to the length for different theories of thermoelasticity. It is observed that the behaviour of temperature change and lateral deflection are different for all thermoelastic theories. The value of temperature change and lateral distribution decreases for small values of length and then remain stationary for higher values. Its value is greater for the three-phase-lag than that of GN-II and GN-III.

Figure 14 shows the displacement component with respect to the length for different thermoelasticity theories. Initially, the value of displacement component is stationary and after that, it increases for the remaining region for all the theories of thermoelasticity. The value of dis-

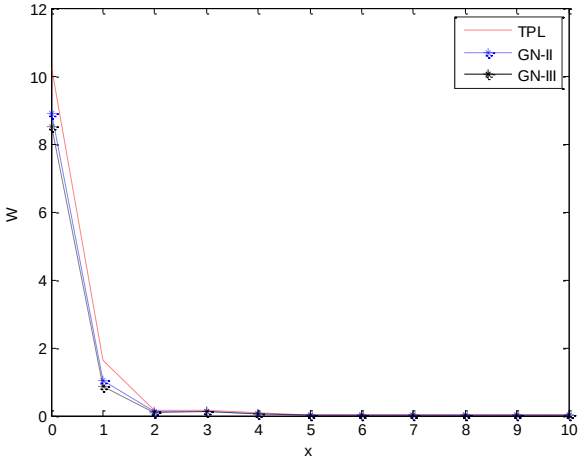

Figure 13: Lateral deflection with respect to the length for different theories of thermoelasticity

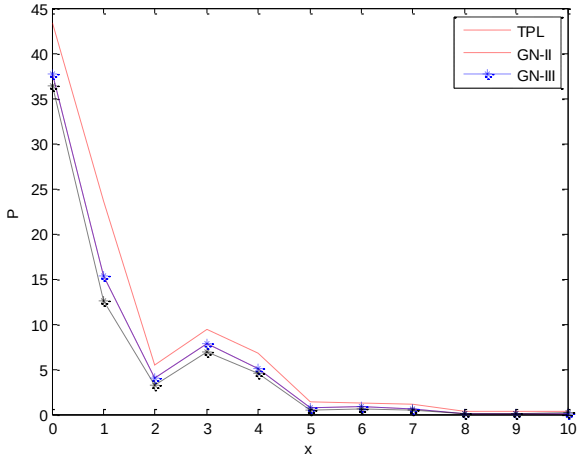

Figure 16: Chemical potential with respect to the length for different theories of thermoelasticity

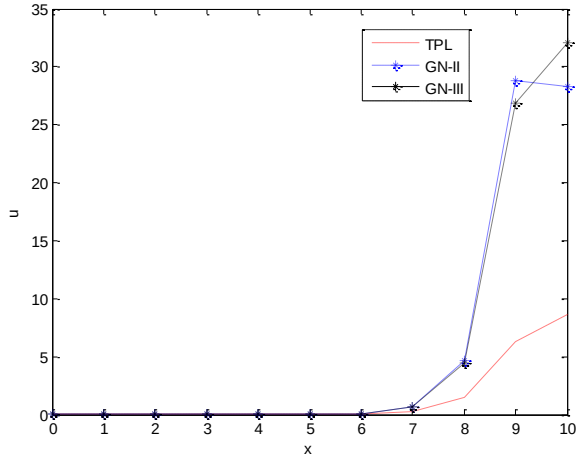

Figure 14: Displacement component with respect to the length for different theories of thermoelasticity

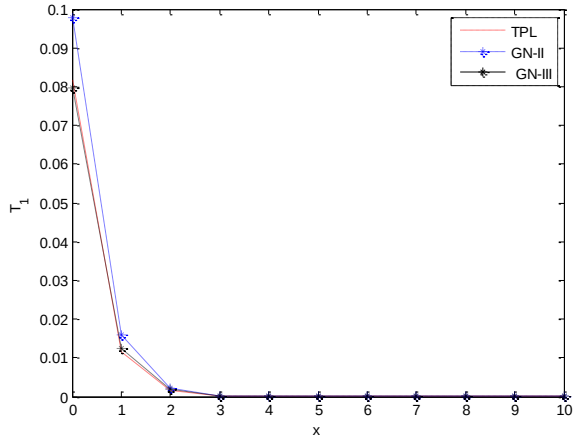

Figure 17: Temperature change with respect to the length for different theories of thermoelasticity

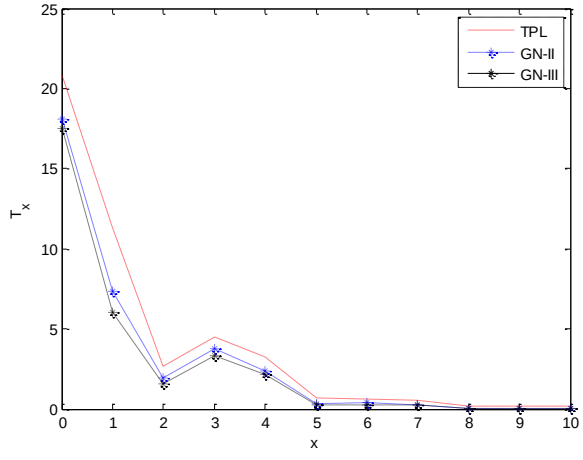

Figure 15: Axial stress with respect to the length for different theories of thermoelasticity

placement component is less for three-phase-lag than that of GN-II and GN-III. Figures 15 and 16 represent the axial stress and chemical potential with respect to the length for different thermoelastic theories. The values of both axial stress and chemical potential are oscillatory for the range $0 \leq x \leq 5$ and then decrease in the remaining range. It is clear from the figure that the variation of axial stress and chemical potential are almost the same for three-phase-lag, GN-II and GN-III, but there are differences between their values. Figure 17 depicts the temperature change with respect to the length for different theories of thermoelasticity. It is noticed that the temperature decreases smoothly for the range $0 \leq x \leq 4$ and then it remains stationary in the assumed region for all the thermoelastic theories. Figure 18 shows the lateral deflection with respect to the length for different theories of thermoelasticity. It is observed that the behaviour and variation is same for all the

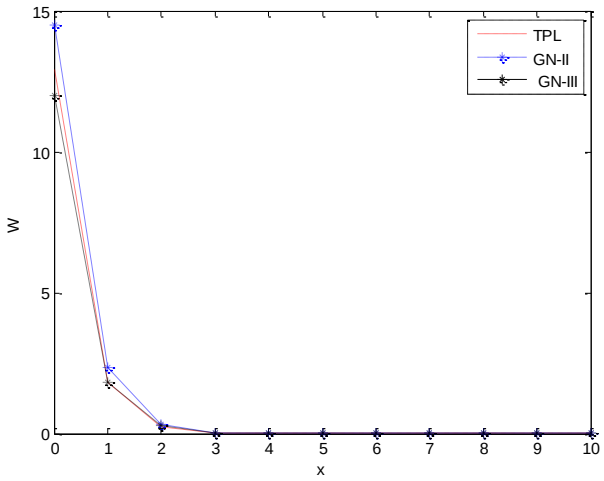

Figure 18: Lateral deflection with respect to the length for different theories of thermoelasticity

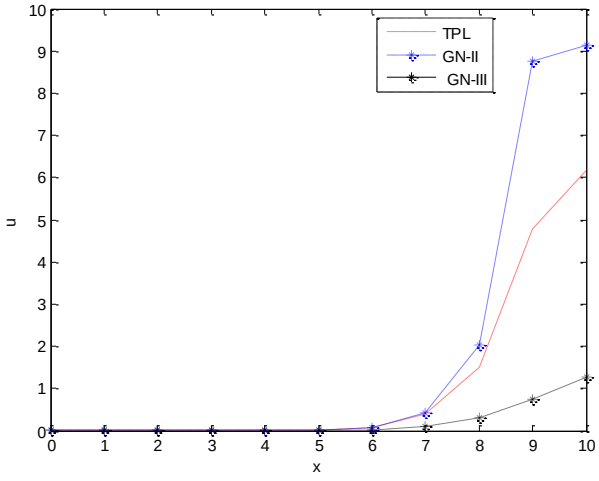

Figure 19: Displacement component with respect to the length for different theories of thermoelasticity

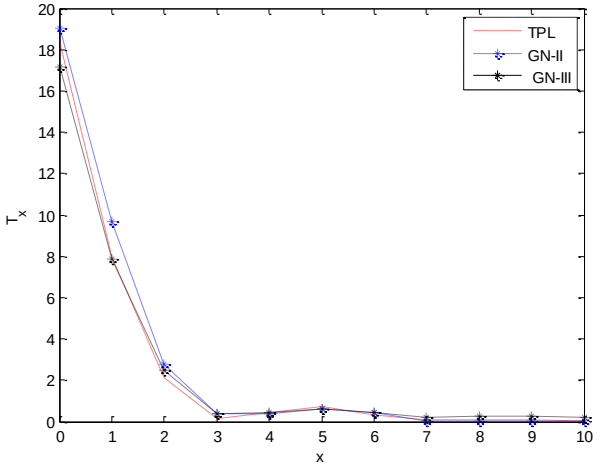

Figure 20: Axial stress with respect to the length for different theories of thermoelasticity

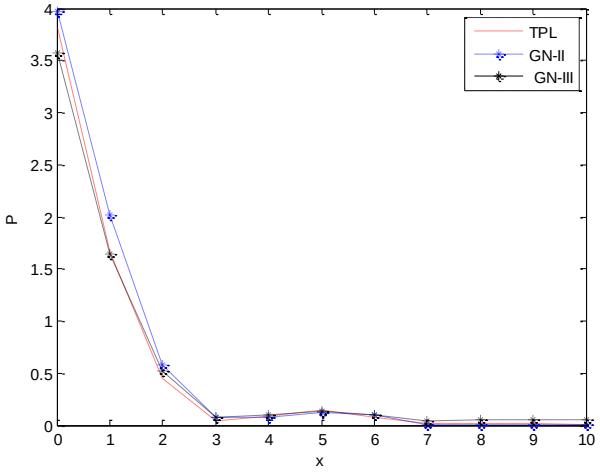

Figure 21: Chemical potential with respect to the length for different theories of thermoelasticity

cases of thermoelastic theories, that is, three-phase-lag, GN-II and GN-III. The value of lateral deflection is more for GN-II than that of three-phase-lag and GN-III.

Figure 19 represents the displacement component with respect to the length for different theories of thermoelasticity. The value of displacement component is stationary for a smaller value of thickness and then increases for higher values of thickness. The displacement component is same for the range $0 \le x \le 5$, but a small difference is observed in the considered region for all the theories of thermoelasticity. Figures 20 and 21 depict the axial stress and chemical potential with respect to the length for different theories of thermoelasticity. The values of axial stress and chemical potential fall down to range $0 \le x \le 3$ and after that decrease smoothly for the remaining region. It is observed that the values of axial stress and chemical potential are greater for GN-II as compared to three-phase-lag and GN-III for the range $0 \le x < 4$ and reversed behaviour is noticed for the assumed region.

9 Conclusions

The thermoelastic nanoscale beam in a modified couple stress thermoelastic in the context of the three-phase-lag diffusion model is studied. The Euler-Bernoulli beam assumption and the Laplace transform technique are used to write the basic set of equations in the form of vector-matrix differential equation, which is then solved by the eigenvalue approach. A numerical technique has been adopted to recover the solutions in the physical domain. Special type of instantaneous and distributed sources are taken

to show the utility of the approach. The general algorithm of the inverse Laplace transform is developed to compute the results numerically. The numerical results are depicted graphically to show the effects of phase lags, with and without energy dissipation on the physical quantities. For thermal source, it is observed from the figures that lateral deflection, axial stress and chemical potential for distributed source are higher in comparison with instantaneous source, whereas opposite behaviour is noticed in the case of temperature change and displacement component for three-phase-lag, GN-II and GN-III theories of thermoelasticity. For chemical potential source, it is noticed that the temperature change, lateral deflection, displacement component, axial stress and chemical potential are greater values for instantaneous source as compared with distributed source for different theories of thermoelasticity, that is, three-phase-lag, Green-Naghdi-II and Green-Naghdi-III. Three-phase-lag model is very helpful in problems like nuclear boiling, exothermic catalytic reactions, phonon-electron interactions and phonon scattering.

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